On the scaling property in fluctuation theory for stable Lévy processes

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Abstract

We find an expression for the joint Laplace transform of the law of $(T_{[x,+\infty[}, X_{T_{[x,+\infty[}})$ for a Lévy process X, where $T_{[x,+\infty[}$ is the first hitting time of $[x,+\infty[$ by X. When X is an α-stable Lévy process, with $1<\alpha<2$, we show how to recover from this formula the law of $X_{T_{[x,+\infty[}}$; this result was already obtained by D. Ray, in the symmetric case and by N. Bingham, in the case when X is non spectrally negative. Then, we study the behaviour of the time of first passage $T_{[x,+\infty[}$ conditioned to $\{X_{T_{[x,+\infty[}} - x \leq h\}$ when h tends to 0. This study brings forward an asymptotic variable T_x^0, which seems to be related to the absolute continuity of the law of the supremum of X.

Key words : Fluctuation theory, Scaling property, Lévy processes, Stable processes, Overshoots, First passage time.

1 Introduction

Let $X = (X_t, t \geq 0)$ denote a Lévy process with characteristic exponent ψ.

For $t \geq 0$ we set :

$$S_t = \sup_{0\leq s\leq t} X_s \quad \text{and} \quad I_t = \inf_{0\leq s\leq t} X_s,$$

and for $x > 0$:

$$T_{[x,+\infty[} = \inf\{s>0 : X_s \geq x\} \text{ and } K_x = X_{T_{[x,+\infty[}} - x.$$

We denote by $\mathbf{e}_\gamma$ an exponential variable with parameter γ, independent of X.

Our interest focusses on the joint law of $(T_{[x,+\infty[}, X_{T_{[x,+\infty[}})$ (see [5]) and on questions concerned with the absolute continuity of the law of S_t.

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In section 2 we present some known results, which we will be needed in the following sections. The first result (Proposition 1) is a famous formula obtained by Pecherskii and Rogozin which concerns the joint Laplace transform of $(T_{[\mathbf{e}_\gamma,+\infty[}, X_{T_{[\mathbf{e}_\gamma,+\infty[}})$. The second result (Proposition 2) deals with the asymptotic behaviour of the quantity $P(S_1 < x)$ in the stable case when x tends to 0.

In section 3 we obtain an expression of the joint Laplace transform of the pair $(T_{[x,+\infty[}, X_{T_{[x,+\infty[}})$. First, we remark that the Laplace transform of this quantity can be expressed by means of the joint Laplace transform of $(T_{[\mathbf{e}_\gamma,+\infty[}, X_{T_{[\mathbf{e}_\gamma,+\infty[}})$, for which we dispose of a formula (Proposition 1). In this way we obtain the result by inverting a Laplace transform.

Then, we pass to the α-stable case, with $1 < \alpha < 2$. By means of the scaling property and of Proposition 2, we prove some asymptotic properties associated to quantities introduced in the previous paragraph. After this, we recover the law of $X_{T_{[x,+\infty[}}$, as a corollary of these properties. This law is already known. In [8] Ray gives an expression of its density in the symmetric stable case. In [2] Bingham generalises this result for non-spectrally negative stable processes. However, the interest of this part of the work is to point out the role of the scaling property in the obtention of this result.

Finally, we focus on the asymptotic behaviour of the random variable $T_{[x,+\infty[}$ conditionned upon $\{K_x \leq h\}$ when h tends to $0+$. We show that a convergence in law holds towards a random variable which we denote T_x^0.

2 Preliminaries

First, we introduce some notations that will be useful afterwards. For $q > 0$, $\operatorname{Re}\lambda \leq 0$ and $\operatorname{Re}\mu \geq 0$, we define :

$$\psi_q^+(\lambda) = \exp\left(- \int_0^{+\infty} (e^{\lambda x} - 1) d_x\Big(\int_0^{+\infty} u^{-1} e^{-qu} P(X_u > x) du \Big)\right), \tag{1}$$

and

$$\psi_q^-(\mu) = \exp\left(\int_{-\infty}^{0} (e^{\mu x} - 1) d_x\Big(\int_0^{+\infty} u^{-1} e^{-qu} P(X_u < x) du \Big)\right). \tag{2}$$

We use the following result due to Pecherskii and Rogozin :

Proposition 1 ([7], p.420)**.** *For all $\gamma, \lambda, \mu > 0$, we have :*

$$E[\exp(-\lambda T_{[\mathbf{e}_\gamma,+\infty[} - \mu K_{\mathbf{e}_\gamma})] = \frac{\gamma}{\gamma - \mu}\left(1 - \frac{\psi_\lambda^+(-\gamma)}{\psi_\lambda^+(-\mu)}\right). \tag{3}$$

Remark 1. It follows from the definition of ψ_γ^+ and ψ_γ^- that :

$$\psi_\gamma^+(i\lambda)\ \psi_\gamma^-(i\lambda) = \frac{\gamma}{\gamma + \psi(\lambda)}. \tag{4}$$

On the other hand, it has been shown by Rogozin in [9] that for $\lambda, \mu \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \leq 0$, $\mathrm{Re}\,\mu \geq 0$, we have :

$$\psi_\gamma^+(\lambda) = E[\exp(\lambda S_{\mathbf{e}_\gamma})] \quad \text{and} \quad \psi_\gamma^-(\mu) = E[\exp(\mu I_{\mathbf{e}_\gamma})]. \tag{5}$$

Hence, the representation of $\gamma(\gamma + \psi(\lambda))^{-1}$ in (4) is an infinitely divisible factorization.

Remark 2. We can find formula (3) in [1] (chap.VI, exercise 1, p.182) expressed as follows :

$$\int_0^{+\infty} e^{-\lambda a} E\left[\exp(-\beta T_{[a,+\infty[} - \theta X_{T_{[a,+\infty[}})\right] da = \frac{\kappa(\beta, \lambda + \theta) - \kappa(\beta, \theta)}{\lambda \kappa(\beta, \lambda + \theta)}, \tag{6}$$

where $\kappa(\cdot, \cdot)$ is defined by :

$$\exp\left(-\kappa(\beta, \theta)\right) = E\left[\exp(-\beta \bar{\tau}_1 - \theta S_{\bar{\tau}_1})\right], \tag{7}$$

where $\bar{\tau}$ denotes the right-continuous inverse of the local time at level 0 of the reflected process $S - X$.

The relation between formulae (3) and (6) is simple to establish. In fact, using the following formula ([1], Chap. VI, Corollary 10, p. 165) :

$$\kappa(\beta, \theta) = \kappa(1, 0) \exp\left(\int_0^{+\infty} dt \int_{[0,+\infty[} (e^{-t} - e^{-\beta t - \theta x}) t^{-1} P(X_t \in dx) \right), \tag{8}$$

we prove that for $q, \lambda > 0$, we have :

$$\psi_q^+(-\lambda) = \frac{\kappa(q, 0)}{\kappa(q, \lambda)}.$$

In the case of a stable process with index α, the following result will play a key role :

Proposition 2 ([1], Chap.VIII, Proposition 2, p.219)**.** *We set $\rho = P(X_1 > 0)$. If $\rho \in]0, 1[$, then there exists $k > 0$ such that :*

$$P(S_1 \leq x) \sim k x^{\alpha\rho} \quad as \quad x \to 0+. \tag{9}$$

3 About the law of $(T_{[x,+\infty[}, X_{T_{[x,+\infty[}})$

3.1 The general case

The purpose of this paragraph is to express the joint Laplace transform of $(T_{[x,+\infty[}, X_{T_{[x,+\infty[}})$ for $x > 0$:

$$E[\exp(-\lambda T_{[x,+\infty[} - \mu X_{T_{[x,+\infty[}})].$$

Formula (3) gives an expression of the Laplace transform of the quantity we are interested in. So the problem is reduced to invert this Laplace transform.

First, we note that for $\gamma > \mu > 0$, we have :

$$\frac{1}{\gamma-\mu} = \int_0^{+\infty} e^{-(\gamma-\mu)x} dx \quad \text{and} \quad \frac{\gamma}{\gamma-\mu} = 1 + \frac{\mu}{\gamma-\mu}.$$

On the other hand, from (5), we obtain for $\gamma, \lambda > 0$:

$$\psi_\gamma^+(-\lambda) = E[\exp(-\lambda S_{\mathbf{e}_\gamma})] = \int_0^{+\infty} \lambda\, e^{-\lambda z} P(S_{\mathbf{e}_\gamma} \leq z)\, dz. \tag{10}$$

As a consequence, (3) can be expressed as :

$$\begin{aligned}
&\int_0^{+\infty} e^{-\gamma x}\, E[\exp(-\lambda T_{[x,+\infty[} - \mu K_x)]\, dx \\
&\qquad = \frac{1}{\gamma-\mu} - \frac{1}{\psi_\lambda^+(-\mu)} \left(\frac{\gamma}{\gamma-\mu} \int_0^{+\infty} e^{-\gamma z} P(S_{\mathbf{e}_\lambda} \leq z) dz \right) \\
&\qquad = \int_0^{+\infty} e^{-\gamma x} \left(e^{\mu x} - \frac{P(S_{\mathbf{e}_\lambda} \leq x)}{\psi_\lambda^+(-\mu)} \right) dx - \frac{I(\gamma,\lambda,\mu)}{\psi_\lambda^+(-\mu)},
\end{aligned} \tag{11}$$

where :

$$I(\gamma,\lambda,\mu) = \mu \int_0^{+\infty} P(S_{\mathbf{e}_\lambda} \leq z) \left(\int_0^{+\infty} e^{-\gamma(y+z)} e^{\mu y} dy \right) dz.$$

Note that thanks to the change of variables $v = z$, $x = y + z$ and Fubini's theorem, we get :

$$I(\gamma,\lambda,\mu) = \mu \int_0^{+\infty} e^{-(\gamma-\mu)x} \left(\int_0^{x} e^{-\mu v} P(S_{\mathbf{e}_\lambda} \leq v) dv \right) dx.$$

If we put this expression for $I(\gamma,\lambda,\mu)$ in (11), we obtain using (10) :

$$\int_0^{+\infty} e^{-\gamma x} E[\exp(-\lambda T_{[x,+\infty[} - \mu K_x)] dx = \frac{1}{\psi_\lambda^+(-\mu)} \int_0^{+\infty} e^{-(\gamma-\mu)x} J(\mu,\lambda;x) dx, \tag{12}$$

where :

$$J(\mu,\lambda;x) = \int_x^{+\infty} \mu e^{-\mu v} P(S_{\mathbf{e}_\lambda} \leq v) dv - e^{-\mu x} P(S_{\mathbf{e}_\lambda} \leq x). \tag{13}$$

Therefore we have :

$$E[\exp(-\lambda T_{[x,+\infty[} - \mu K_x)] = \frac{e^{\mu x}}{E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right]} J(\mu,\lambda;x). \tag{14}$$

Observe that :

$$\begin{aligned} J(\mu,\lambda;x) &= E\left[\int\limits_{x\vee S_{\mathbf{e}_\lambda}}^{+\infty} \mu e^{-\mu v}dv\right] - e^{-\mu x}P(S_{\mathbf{e}_\lambda}\le x) \\ &= E\left[e^{-\mu(x\vee S_{\mathbf{e}_\lambda})}\right] - e^{-\mu x}P(S_{\mathbf{e}_\lambda}\le x), \end{aligned}$$

and thus :

$$J(\mu,\lambda;x) = E\left[e^{-\mu S_{\mathbf{e}_\lambda}} 1_{\{S_{\mathbf{e}_\lambda}\ge x\}}\right]. \tag{15}$$

Theorem 3. *For all $\lambda,\mu>0$ and $x\ge 0$:*

$$E[\exp(-\lambda T_{[x,+\infty[} - \mu X_{T_{[x,+\infty[}})] = \frac{E\left[e^{-\mu S_{\mathbf{e}_\lambda}} 1_{\{S_{\mathbf{e}_\lambda}\ge x\}}\right]}{E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right]}. \tag{16}$$

Proof. It is immediate from (14) and (15). □

3.2 The stable case

Henceforth, X will denote a real-valued stable Lévy process with index $\alpha\in]1,2]$. We denote :

$$\rho = P(X_1\ge 0).$$

We know that in this case, $\rho\in[1-1/\alpha, 1/\alpha]$ (see [1]), in particular $\rho\in]0,1[$. The cases $\rho = 1-1/\alpha$ and $\rho = 1/\alpha$ correspond to the cases when X has no negative jumps (spectrally positive case) and no positive jumps (spectrally negative case), respectively.

3.2.1 Scaling for $S_{\mathbf{e}_\lambda}$ and some asymptotic results

In Theorem 3, we see a link between the random variable $S_{\mathbf{e}_\lambda}$ and the joint distribution of $\left(T_{[x,+\infty[}, X_{T_{[x,+\infty[}}\right)$. In this section, we show how the scaling property allows to study the absolute continuity of the law of $S_{\mathbf{e}_\lambda}$ and the asymptotic behavior of some quantities associated to this random variable.

Proposition 4. *For all $\lambda>0$, the law of the random variable $S_{\mathbf{e}_\lambda}$ is absolutely continuous. Its density f_λ can be expressed as :*

$$f_\lambda(x) = \frac{\lambda\alpha}{x}E\left[T_{[x,+\infty[}\exp(-\lambda T_{[x,+\infty[})\right], \qquad x>0. \tag{17}$$

Proof. If we take the limit when μ goes to $0+$ in (16), we get :

$$P(S_{\mathbf{e}_\lambda}\le x) = 1 - E\left[\exp(-\lambda T_{[x,+\infty[})\right].$$

On the other hand, the scaling property yields :

$$E\left[\exp(-\lambda T_{[x,+\infty[})\right] = E\left[\exp(-\lambda x^\alpha T_{[1,+\infty[})\right].$$

The result follows. □

Lemma 5 (Scaling)**.** *For all $\lambda, \mu > 0$ and $x \geq 0$:*

(i) $P(S_{\mathbf{e}_\lambda} \leq x) = P\left(S_{\mathbf{e}} \leq x\lambda^{1/\alpha}\right) = \int_0^{+\infty} e^{-v} P\left(S_1 \leq \frac{x\lambda^{1/\alpha}}{v^{1/\alpha}}\right) dv.$

(ii) $E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right] = E\left[e^{-\frac{\mu}{\lambda^{1/\alpha}} S_{\mathbf{e}}}\right] = \int_0^{+\infty} e^{-v} P\left(S_{\mathbf{e}} \leq \frac{v\lambda^{1/\alpha}}{\mu}\right) dv.$

Proof.
(i) We have for any $\lambda, \mu > 0$ and $x \geq 0.$:

$$\begin{aligned} P(S_{\mathbf{e}_\lambda} \leq x) &= \int_0^{+\infty} \lambda e^{-\lambda u} P(S_u \leq x) du \\ \text{(change of variables } \lambda u = v) &= \int_0^{+\infty} e^{-v} P(S_{v/\lambda} \leq x) dv \\ \text{(scaling)} &= \int_0^{+\infty} e^{-v} P\left(S_1 \leq \frac{x\lambda^{1/\alpha}}{v^{1/\alpha}}\right) dv, \end{aligned}$$

which proves (i).

(ii) From (10), (i) and the change of variables $\mu z = v$:

$$E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right] = \int_0^{+\infty} \mu e^{-\mu z} P\left(S_{\mathbf{e}} \leq z\lambda^{1/\alpha}\right) dz = \int_0^{+\infty} e^{-v} P\left(S_{\mathbf{e}} \leq \frac{v\lambda^{1/\alpha}}{\mu}\right) dv,$$

which completes the proof. □

Proposition 6. *There exists a constant $k^* > 0$ such that :*

(i) $P(S_{\mathbf{e}_\lambda} \leq x) \sim k^* \, x^{\alpha\rho} \, \lambda^\rho$ *when* $\lambda \to 0+$.

(ii) $\int_x^{+\infty} \mu e^{-\mu v} P(S_{\mathbf{e}_\lambda} \leq v) dv \sim k^* \left(\int_{\mu x}^{+\infty} e^{-y} y^{\alpha\rho} dy\right) \mu^{-\alpha\rho} \lambda^\rho$ *when* $\lambda \to 0+$.

(iii) $E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right] \sim k^* \, \Gamma(1+\alpha\rho) \, \mu^{-\alpha\rho} \, \lambda^\rho$ *when* $\lambda \to 0+$.

(iv) $E\left[e^{-\mu S_{\mathbf{e}_\lambda}} 1_{\{S_{\mathbf{e}_\lambda} \geq x\}}\right] \sim k^* \alpha\rho \left(\int_{\mu x}^{+\infty} e^{-y} y^{\alpha\rho - 1} dy\right) \mu^{-\alpha\rho} \lambda^\rho$ *when* $\lambda \to 0+$.

Proof.
(i) Thanks to Proposition 2 and Lemma 5 (i), the dominated convergence theorem yields :

$$\frac{P(S_{\mathbf{e}_\lambda} \leq x)}{\lambda^\rho} = \int_0^{+\infty} e^{-v} \frac{P\left(S_1 \leq \frac{x\lambda^{1/\alpha}}{v^{1/\alpha}}\right)}{\lambda^\rho} dv \underset{\lambda \to 0+}{\longrightarrow} k \, \Gamma(1-\rho) \, x^{\alpha\rho}.$$

It suffices to choose $k^* = k \, \Gamma(1-\rho)$ to finish the proof of (i).

(ii) Applying the dominated convergence theorem, we deduce from (i) :

$$\frac{\int_x^{+\infty} \mu e^{-\mu v} P(S_{\mathbf{e}_\lambda} \leq v) dv}{\lambda^\rho} \underset{\lambda \to 0+}{\longrightarrow} k^* \int_x^{+\infty} \mu e^{-\mu v} v^{\alpha\rho} dv.$$

we obtain (ii) by the change of variables $y = \mu v$.

(iii) It suffices to consider $x = 0$ in assertion (ii).

(iv) Using formulae (13) and (15), (iv) is a consequence of (i) and (ii). □

In [8], Ray gives an expression for the density of the law of $X_{T_{[x,+\infty[}}$ in the symmetric case. In [2], Bingham generalizes this result to the case when X is not spectrally negative. Now, we recover this result as a corollary of Theorem 3 and Proposition 6 (see also Lemma 4.1 in [10]).

Corollary 7 ([8], [2])**.** *If $\alpha\rho < 1$, then :*

$$X_{T_{[x,+\infty[}} \overset{(loi)}{=} \frac{x}{\beta_{\alpha\rho, 1-\alpha\rho}}. \tag{18}$$

Consequently :

$$P(X_{T_{[x,+\infty[}} \in dy) = \rho(x,y)dy, \text{ when } \rho(x,y) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{1}{y} \left(\frac{x}{y-x}\right)^{\alpha\rho} 1_{]x,\infty[}(y).$$

Proof. If we let λ tend to $0+$ in (16), we get with the help of Proposition 6,

$$E[\exp(-\mu X_{T_{[x,+\infty[}})] = \frac{1}{\Gamma(\alpha\rho)} \int_{\mu x}^{+\infty} e^{-y} y^{\alpha\rho-1} dy,$$

and by using the identity $y^{\alpha\rho-1} = \frac{1}{\Gamma(1-\alpha\rho)} \int_0^{+\infty} e^{-yv} v^{-\alpha\rho} dv$, we obtain :

$$\begin{aligned}
E[\exp(-\mu X_{T_{[x,+\infty[}})] &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_{\mu x}^{+\infty} dy \int_0^{+\infty} e^{-y(v+1)} v^{-\alpha\rho} dv \\
&= \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^{+\infty} \frac{e^{-\mu x(v+1)}}{v+1} v^{-\alpha\rho} dv \quad \text{(Fubini's theorem)} \\
&= \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^1 e^{-\mu x/z} z^{\alpha\rho-1} (1-z)^{-\alpha\rho} dz \quad \text{(change of} \\
&\qquad\qquad \text{variables } z = 1/(v+1)),
\end{aligned}$$

which proves Corollary 7. □

Remark 3. Similarly, the law of K_x is absolutely continuous. Its density is given by :

$$\rho_K(x,y) = \rho(x, x+y) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{1}{(y+x)} \left(\frac{x}{y}\right)^{\alpha\rho} 1_{]0,\infty[}(y).$$

Remark 4. This result has been generalized in the papers [4] and [6].

3.2.2 An asymptotic law

Note first that a standard application of the Markov property permits to show that :

$$P(S_t \in]\,x, x{+}h\,]) = \iint_{[0,t[\times]x,x+h]} P(T_{[x+h-y,+\infty[} \geq t{-}s)\, P(T_{[x,+\infty[} \in ds, X_{T_{[x,+\infty[}} \in dy).$$

Taking the limit in this equality when h tends to 0, we may expect to find links between the absolute continuity of the law of S_t and the asymptotic behavior of the random variable $T_{[x,+\infty[}$ conditioned to $\{K_x \leq h\}$.

With this motivation, we introduce for each $x, h > 0$ the random variable T_x^h whose law is given by :

$$P(T_x^h \in \cdot\ ) = P(T_{[x,+\infty[} \in \cdot\ \mid K_x \leq h).$$

Thus the aim in this section is to study the asymptotic behavior of the variables T_x^h when h goes to 0+. For this, we start by computing the asymptotic probability of the events $\{K_x \leq h\}$.

Lemma 8. *For all $x > 0$:*

$$\frac{P(K_x \leq h)}{h^{1-\alpha\rho}} \underset{h\to 0+}{\longrightarrow} \frac{\sin(\pi\alpha\rho)}{\pi(1-\alpha\rho)}\, x^{\alpha\rho-1}. \tag{19}$$

Proof. By Corollary 7, we have :

$$\begin{aligned} P(X_{T_{[x,+\infty[}} \leq x + h) &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_x^{x+h} \frac{1}{y}\left(\frac{x}{y-x}\right)^{\alpha\rho} dy \\ &= \frac{\sin(\pi\alpha\rho)}{\pi}\ x^{\alpha\rho} h^{1-\alpha\rho} \int_0^1 \frac{1}{(uh+x)u^{\alpha\rho}} du \quad \text{(change of} \\ &\qquad\qquad\qquad\qquad \text{variables } u = (y-x)/h). \end{aligned}$$

The result follows by dividing both sides by $h^{1-\alpha\rho}$ and applying the dominated convergence theorem. ☐

Now, we define the probability measure $P_\lambda^{(x)}$ by :

$$P_\lambda^{(x)}(A) = \frac{E\left[1_A \exp(-\lambda T_{[x,+\infty[})\right]}{E\left[\exp(-\lambda T_{[x,+\infty[})\right]}, \quad A \in \mathcal{F}_\infty.$$

The following lemma is the analogue of Lemma 8 for the probability measure $P_\lambda^{(x)}$.

Lemma 9. *For all $\lambda, x > 0$, we have :*

$$\frac{P_\lambda^{(x)}(X_{T_{[x,+\infty[}} - x \leq h)}{h^{1-\alpha\rho}} \underset{h\to 0+}{\longrightarrow} \frac{\sin(\pi\alpha\rho)}{k^*\pi\alpha\rho(1-\alpha\rho)}\ \frac{\lambda^{-\rho} f_\lambda(x)}{P(S_{\mathbf{e}_\lambda} \geq x)}, \tag{20}$$

where k^* is the constant appearing in Proposition 6.

Proof. We consider the fonction $U : [0,+\infty[\to [0,+\infty[$ defined by :

$$U(h) = P_{\lambda}^{(x)}\left(K_x \le h\right).$$

By Tauberian theorem (see p.10 in [1]), the behavior of U around $0+$ is related to the behavior of its Laplace transform at infinity.

We remark that :

$$\int_0^{+\infty} e^{-\mu y} U(dy) = E_{\lambda}^{(x)}\left[\exp(-\mu K_x)\right] = \frac{E\left[\exp(-\lambda T_{[x,+\infty[} - \mu K_x)\right]}{E\left[\exp(-\lambda T_{[x,+\infty[})\right]}. \tag{21}$$

On the other hand, thanks to (13), (14) and an obvious change of variables, we obtain :

$$E\left[\exp(-\lambda T_{[x,+\infty[} - \mu K_x)\right] = \frac{\int_0^{+\infty} e^{-y}\left(P\left(S_{\mathbf{e}_\lambda} \le \frac{y}{\mu} + x\right) - P\left(S_{\mathbf{e}_\lambda} \le x\right)\right) dy}{E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right]}. \tag{22}$$

Using Lemma 4 and applying the dominated convergence theorem, we show that :

$$\mu \int_0^{+\infty} e^{-y}\left(P\left(S_{\mathbf{e}_\lambda} \le \frac{y}{\mu} + x\right) - P\left(S_{\mathbf{e}_\lambda} \le x\right)\right) dy \underset{\mu\to+\infty}{\longrightarrow} f_\lambda(x). \tag{23}$$

We know from part (ii) of Lemma 5 that :

$$E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right] = E\left[e^{-\frac{1}{\lambda^{1/\alpha}} S_{\mathbf{e}_{\mu^{-\alpha}}}}\right],$$

and therefore, from part (iii) of Proposition 6, we obtain that :

$$\mu^{\alpha\rho} E\left[e^{-\mu S_{\mathbf{e}_\lambda}}\right] \underset{\mu\to+\infty}{\longrightarrow} k^*\Gamma(1+\alpha\rho)\lambda^{\rho}. \tag{24}$$

Thus, from (21), (22), (23) and (24), we get :

$$\mu^{1-\alpha\rho} \int_0^{+\infty} e^{-\mu y} U(dy) \underset{\mu\to+\infty}{\longrightarrow} \frac{1}{k^*\Gamma(1+\alpha\rho)} \frac{\lambda^{-\rho} f_\lambda(x)}{P(S_{\mathbf{e}_\lambda} \ge x)}, \tag{25}$$

and then, thanks to the Tauberian theorem (see p.10 in [1]), we obtain :

$$\frac{1}{h^{1-\alpha\rho}} U(h) \underset{h\to 0+}{\longrightarrow} \frac{1}{k^*\Gamma(1+\alpha\rho)\Gamma(2-\alpha\rho)} \frac{\lambda^{-\rho} f_\lambda(x)}{P(S_{\mathbf{e}_\lambda} \ge x)}, \tag{26}$$

which completes the proof. □

Proposition 10. *For every $\lambda, x > 0$, we have :*

$$E\left[\exp(-\lambda T_{[x,+\infty[}\,) \mid K_x \le h\right] \underset{h\to 0+}{\longrightarrow} \frac{1}{k^*\alpha\rho}\, x^{1-\alpha\rho}\, \lambda^{-\rho}\, f_\lambda(x). \tag{27}$$

Proof. We remark that :

$$E\left[\exp(-\lambda T_{[x,+\infty[}\,) \mid K_x \leq h\right] = \frac{P_\lambda^{(x)}\left(K_x \leq h\right)}{P\left(K_x \leq h\right)} E\left[\exp(-\lambda T_{[x,+\infty[})\right].$$

Now, the result is a consequence of Lemmas 8 and 9. $\square$

Lemma 11. *For all $x > 0$:*

$$\lim_{\lambda \to 0+} \frac{1}{k^* \alpha\rho}\, x^{1-\alpha\rho}\, \lambda^{-\rho}\, f_\lambda(x) = 1. \tag{28}$$

Proof. The scaling property entails :

$$E[T_{[x,+\infty[}\exp(-\lambda T_{[x,+\infty[})] = \int_0^{+\infty} e^{-v}(1-v) P\left(S_1 \leq \frac{x\lambda^{1/\alpha}}{v^{1/\alpha}}\right) dv,$$

and then, by definition of f_λ we obtain from Proposition 2 and the dominated convergence theorem that :

$$\begin{aligned} \lambda^{-\rho}\, f_\lambda(x) \underset{\lambda \to 0+}{\longrightarrow} k\,\alpha\, x^{\alpha\rho-1} \int_0^{+\infty} e^{-v}\,(1-v)\, v^{-\rho}\, dv \\ = k\,\alpha\,\rho\,\Gamma(1-\rho)\, x^{\alpha\rho-1}. \end{aligned}$$

The result follows. $\square$

Theorem 12. *For each $x > 0$, the family of random variables $\{T_x^h\}_{h>0}$ converges in law as h tends to $0+$. The limit is denoted by T_x^0 and its law is given by :*

$$P\left(T_x^0 \leq t\right) = \frac{\sin(\pi\rho)}{k\,\pi\rho}\, x^{-\alpha\rho} E\left[\frac{T_{[x,+\infty[}\; 1_{\{T_{[x,+\infty[} \leq t\}}}{\left(t - T_{[x,+\infty[}\right)^{1-\rho}}\right].$$

Proof. For $x, h > 0$, we denote by $\mathcal{L}_h^x$ the Laplace transform of the random variable T_x^h and by $\mathcal{L}^x$ the fonction defined by :

$$\mathcal{L}^x(\lambda) = \frac{1}{k^* \alpha\rho}\, x^{1-\alpha\rho}\, \lambda^{-\rho}\, f_\lambda(x).$$

According to Proposition 10, the Laplace transform $\mathcal{L}_h^x$ converges pointwise to the fonction $\mathcal{L}^x$. Since we have already shown in Lemma 11 that :

$$\lim_{\lambda \to 0+} \mathcal{L}^x(\lambda) = 1,$$

it appears that the fonction $\mathcal{L}^x$ is the Laplace transform to some probability measure with support in $\mathbb{R}_+$ (see [3], Theorem 6.6.3, p.190). We denote this limit measure by ν_x.

We remark that :

$$\frac{\mathcal{L}^x(\lambda)}{\lambda} = \frac{1}{\lambda} \int_{[0,+\infty]} e^{-\lambda y} \nu_x(dy) = \int_0^{+\infty} e^{-\lambda y} \nu_x([0,y])\, dy. \tag{29}$$

On the other hand, we have :

$$\frac{\mathcal{L}^x(\lambda)}{\lambda} = \frac{1}{k^*\rho}\, x^{-\alpha\rho}\, \lambda^{-\rho} E\left[T_{[x,+\infty[} \exp(-\lambda T_{[x,+\infty[})\right].$$

By using the identity $\lambda^{-\rho} = \frac{1}{\Gamma(\rho)} \int\limits_0^{+\infty} e^{-\lambda z} z^{\rho-1} dz$, we get :

$$\frac{\mathcal{L}^x(\lambda)}{\lambda} = \frac{\sin(\pi\rho)}{k\,\pi\rho}\, x^{-\alpha\rho} \int\limits_0^{+\infty} e^{-\lambda u} E\left[\frac{T_{[x,+\infty[}\, 1_{\{T_{[x,+\infty[}\leq u\}}}{\left(u - T_{[x,+\infty[}\right)^{1-\rho}}\right] du. \qquad (30)$$

The result is obtained by comparing (29) and (30). $\square$

3.2.3 The laws of $T_{[x,+\infty[}$ and T_x^0 have no point masses

In [7] (Lemma 1), it has been demonstrated in a more general framework that the law of S_t admits no point masses, i.e., for any $x \geq 0$, $P(S_t = x) = 0$. Since we have $\{S_t \geq x\} = \{T_{[x,+\infty[} \leq t\} \cup \{S_t = x\}$, then :

$$P(S_t \geq x) = P(T_{[x,+\infty[} \leq t).$$

So, by using the scaling property, we obtain in particular :

$$S_1^{-\alpha} \overset{(loi)}{=} T_{[1,+\infty[}.$$

Thus we see that $T_{[1,+\infty[}$ admits no point masses either (the same for $T_{[x,+\infty[}$, by scaling). In the following proposition, we find this result directly from the scaling property.

Proposition 13. *For every $x, t > 0$:*

$$P(T_{[x,+\infty[} = t) = P(T_x^0 = t) = 0.$$

Proof. We first show the result for $T_{[x,+\infty[}$. By scaling property, with no loss of generality, we may suppose that $x = 1$.

Suppose, by contradiction, that there is $t_0 > 0$ such that :

$$\delta := P(T_{[1,+\infty[} = t_0) > 0.$$

Suppose now that there is $h_0 > 0$ satisfying :

$$\eta := P(T_{[1,+\infty[} = t_0,\, K_1 > h_0) > 0.$$

In this case, we have :

$$P(\forall u \in [1, 1 + h_0/2] \,:\, T_{[u,+\infty[} = t_0) > \eta,$$

and therefore, for every $u \in [1, 1 + h_0/2]$, we have $P(T_{[u,+\infty[} = t_0) > \eta$. Thus, the scaling property yields that for each $u \in [1, 1 + h_0/2]$:

$$P(T_{[1,+\infty[} = t_0/u^\alpha) > \eta.$$

In other words, for all $s \in [(2/(2+h_0))^\alpha t_0, t_0]$:

$$P(T_{[1,+\infty[} = s) > \eta,$$

which can not be true. We have therefore, for any $h > 0$:

$$P(T_{[1,+\infty[} = t_0,\, K_1 > h) = 0.$$

Thus, for every $h > 0$:

$$0 < \delta = P(T_{[1,+\infty[} = t_0) = P(T_{[1,+\infty[} = t_0,\, K_1 \le h) \le P(K_1 \le h),$$

which is a contradiction, because the right-hand side converges to 0 when h tends to 0.

The result for T^0_x can be obtained from the result for $T_{[x,+\infty[}$, since thanks to Theorem 12 we have :

$$P(t - h < T^0_x \le t) \le \frac{\sin(\pi\rho)}{k\,\pi\rho}\, x^{-\alpha\rho} E\left[\frac{T_{[x,+\infty[}\ 1_{\{t-h<T_{[x,+\infty[}\le t\}}}{\left(t - T_{[x,+\infty[}\right)^{1-\rho}}\right].$$

□

Comment. In a forthcoming publication, I would like to establish more precise links between the absolute continuity of the law of S_t and the asymptotic random variable T^0_x.

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